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Simple Random Walks on Trees

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Let T be a locally finite, infinite tree. The simple random walk on T is the Markov chain in which transition from one vertex v to another vertex w occurs with probability $1/d(v)$ ($d(v)=$ degree of v) if w is adjacent to v, and probability 0 otherwise. We study recurrence properties (no infinite tree is positive recurrent) and relations between the simple random walk and the tree structure, e.g. R-recurrence and R-transience, the action of a random walk on the branches and ends, the growth of a tree.

1. Introduction

Let T be a tree with an infinite (countable) number of vertices and denote by $d(v)$ the degree of the vertex v (i.e. the number of vertices adjacent to v). If T is locally finite ($d(v)<\infty$ for all vertices v) then it makes sense to consider the Markov chain on T with transition probabilities

$$p(v, w)=\begin{cases} \dfrac{1}{d(v)} & \text{if } v, w \text{ are adjacent vertices,}\\ 0 & \text{otherwise.}\end{cases}$$

This will be called the *simple random walk on T* (abbreviated by SRW). Of course this definition makes sense too for every locally finite graph.

The SRW on a tree (or graph) is a natural notion and it is interesting to relate geometric (topological) properties to probabilistic ones. This leads sometimes to new insights and to interesting problems.

There are a few papers where random walks on particular classes of graphs were studied. Some aspects of random walks on finite graphs are considered in [7], [10]. In [2] harmonic functions on trees are treated in great detail. There are many results for random walks on (discrete) groups. Since a discrete group can be represented by its *Cayley*-graph, these results can be interpreted as properties of random walks on certain very regular graphs. In particular free groups correspond to homogeneous trees (all vertices have the same (even) degree); results in this area can be found e.g. in [4], [11], [13].

For standard notions about Markov chains and random walks see [3].

In this paper we investigate recurrence properties of trees (Section 2, Prop. 3.1) and relations between the radius of convergence (spectral radius) of the SRW on a tree T with the growth of T (Section 4) and with the ends of T (Section 3).

To facilitate the exposition we list now some properties of a tree $T=(V, E)$ (most of which will be assumed in the sequel):

(a) the vertex set V is countably infinite,
(b) T is (connected and) locally finite,
(c) $d(v)\geq 2$ for all vertices $v\in V$,
(d) $d(v)\leq D<\infty$ for all vertices $v\in V$.

2. Recurrence

As usual we introduce n-step transition probabilities for the SRW on a tree $T=(V, E)$. For vertices $v, w\in V$ and a natural number n let
$p^n(v, w)=$ probability to be in w at the nth step after starting in v,

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$f^n(v, w)$ = probability to be for the first time in w at the nth step after starting in v, and $p^0(v, v) = 1$, $f^0(v, w) = 0$. Obviously $p^n(v, v) > 0$ and $f^n(v, v) > 0$ iff n is even. We will further use the generating functions

$$G_v(z) = \sum_{n=0}^{\infty} p^{2n}(v, v) z^n$$

and

$$F_v(z) = \sum_{n=1}^{\infty} f^{2n}(v, v) z^n \qquad (v \in V).$$

Then

$$G_v(z) = \frac{1}{1 - F_v(z)}.$$

By [12] all the power series $G_v(z)(v \in V)$ have the same radius of convergence $R = R(T) \geq 1$ (R does not depend on $v \in V$). Since by definition

$$d(v)p(v, w) = d(w)p(w, v)$$

for arbitrary vertices v, w the SRW on T is *reversible*. Therefore we have from [9, Th. 6.1]

LEMMA 2.1. *The sequence* $p^{2n+2}(v, v)/p^{2n}(v, v)$ $(n = 1, 2, \ldots)$ *is increasing and has limit* $1/R$.

As usual we call the SRW on T (or, for short, the tree T):

R-transient, if $G_v(R) < \infty$ $(\Leftrightarrow F_v(R) < 1)$,

R-recurrent, if $G_v(R) = \infty$ $(\Leftrightarrow F_v(R) = 1)$.

An R-recurrent tree is called:

R-null recurrent, if $F'_v(R) = \sum_{n=1}^{\infty} n f^{2n}(v, v) R^{n-1} = \infty$,

R-positive recurrent, if $F'_v(R) < \infty$.

(Strictly speaking to get the usual terminology one should replace R by $\sqrt{R}$). All these properties are independent of $v \in V$ (see [12]). If $R(T) = 1$ then R-transient, R-null recurrent and R-positive recurrent have the usual meaning and probabilistic interpretation and we omit in this case the letter R.

THEOREM 2.2. *The SRW on a tree with properties* (*a*) *and* (*b*) *is never positive recurrent.*

PROOF. Let us assume that the SRW on the tree T is recurrent. Then [3] all positive harmonic functions h are constant, i.e.

$$\left.\begin{aligned} \sum_{w \in V} p(v, w)h(w) = h(v) \\ h \geq 0 \end{aligned}\right\} \Rightarrow h(v) = c, \qquad \forall v \in V.$$

Now consider any (positive) regular measure m,

$$\sum_{v \in V} m(v)p(v, w) = m(w).$$

Since the SRW is reversible

$$d(v)p(v, w) = d(w)p(w, v),$$

we get

$$\sum_{v \in V} \frac{m(v)}{d(v)} p(w, v) = \frac{m(w)}{d(w)}.$$

But this means that $m(\cdot)/d(\cdot)$ is a positive harmonic function, so it is constant

$$\frac{m(v)}{d(v)} = c > 0, \qquad \forall v \in V.$$

Therefore by (a) and (b)

$$\sum_{v \in V} m(v) = c \sum_{v \in V} d(v) = \infty$$

for every (positive) regular measure m. This implies that the (recurrent) SRW is null recurrent (and cannot be positive recurrent).

COROLLARY 2.3. *On any tree with* (a) *and* (b) *the SRW has the property*:

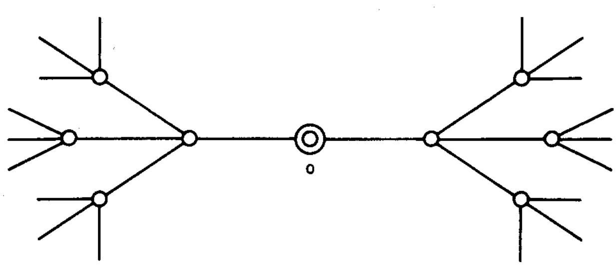
$$\lim_{n \to \infty} p^{2n}(v, v) = 0, \qquad \textit{for all vertices } v \in V.$$

For random walks on infinite discrete groups ($\equiv$ on *Cayley* graphs) more is known, namely: No (irreducible) random walk on an infinite discrete group is R-positive recurrent (where $R^{-1} = \limsup_{n \to \infty} (p^{2n}(v, w))^{1/n}$ as above) [5]. However this property is no more true for trees. This will be shown in the following

EXAMPLE 2.4. (For details of the calculations see [6]).

(a) Consider the following tree T_a: There is a vertex o (origin) such that $d(\mathrm{o}) = 2$ and $d(v) = 4$ whenever $\mathrm{o} \neq v \in V$.

T_a:



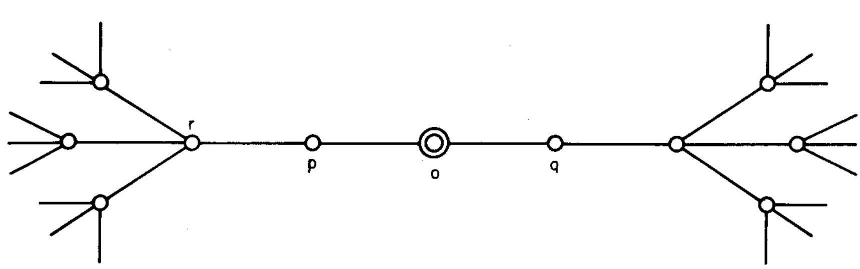
Then

$$F_o(z) = \tfrac{1}{3}(2 - \sqrt{4 - 3z}),$$

$R = R(T_a) = 4/3$ and $F_o(R) < 1$. Therefore T_a is R-transient.

(b) Consider the following tree T_b: $d(\mathrm{o}) = d(\mathrm{p}) = d(\mathrm{q}) = 2$, $d(v) = 4$ whenever $v \in V$, $v \neq \mathrm{o}, \mathrm{p}, \mathrm{q}$.

T_b:



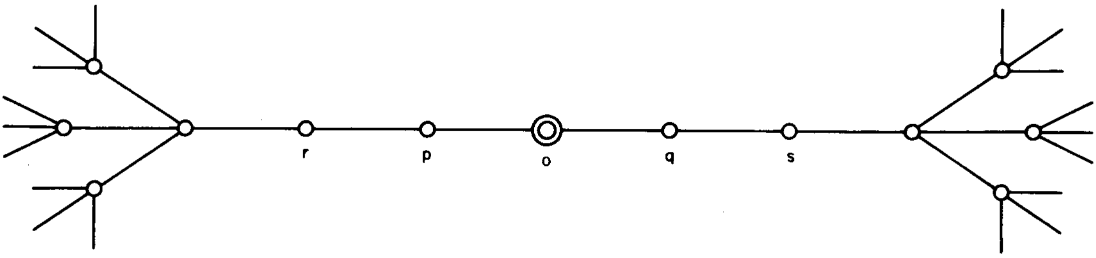
Then

$$F_o(z) = 3z(4+\sqrt{4-3z})^{-1},$$

$R = R(T_b) = 4/3$, but $F_o(R) = 1$, $F_o'(R) = \infty$. Therefore T_b is R-null recurrent.

(c) Consider the following tree T_c: $d(o) = d(p) = d(q) = d(r) = d(s) = 2$, $d(v) = 4$ whenever $v \in V$ and $v \neq o, p, q, r, s$.

T_c:



Then

$$F_o(z) = z(4+\sqrt{4-3z})(8-3z+2\sqrt{4-3z})^{-1},$$

the radius of convergence of $F_o(z)$ is $4/3$, but $F_o(4/3) > 1$. Therefore $1 < R = R(T_c) < 4/3$, where this R is the smallest positive solution of $F_o(z) = 1$ (actually $R = 2\sqrt{13}-6$). Since $z = R$ is a regular point for F_o we have $F_o'(R) < \infty$. Therefore T_c is R-positive recurrent. This is equivalent to

$$0 < \lim_{n\to\infty} R^n p^{2n}(v, v) < \infty, \qquad \text{for all } v \in V.$$

(d) Continuing in this way we obtain trees T_d, T_e, $T_f, \dots$ which are all R-positive recurrent (R is always >1 and depends on the tree).

3. Branches and Ends

We consider trees $T = (V, E)$ with properties (a), (b) and (c) (no dead ends). An infinite path $[p_1, p_2, \dots]$ is a subtree of T with distinct vertices p_i such that $[p_i, p_{i+1}] \in E$. Two infinite paths are equivalent if their vertex sets differ only by finitely many elements. An equivalence class of infinite paths is called an *end* b of T. The set $B(T)$ of all ends of T forms in a natural way the boundary of (the SRW on) T [2].

Every vertex u gives T an orientation: An edge $[v, w]$ is oriented from v to w if the shortest path between u and w passes through v. More generally we write $v \overset{u}{\to} w$ if the shortest path between u and w passes through v. If $[u, v] \in E$ then the *branch* at u with root $[u, v]$, denoted by $B_{u,v}$, is the subtree of T spanned by u, v and all vertices w such that $v \overset{u}{\to} w$.

Each branch $B_{u,v}$ gives rise to a SRW whose transition probabilities coincide with those of the SRW on T with one exception:

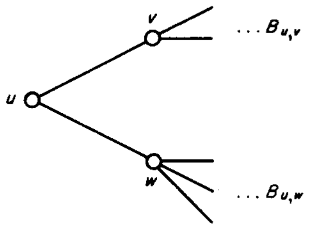
$$p(u, v) = \frac{1}{d(u)} \quad \text{for the SRW on } T\text{, but}$$

$$p(u, v) = 1 \qquad \text{for the SRW on } B_{u,v}.$$

We write $G_{u,v}(z)$ (resp. $F_{u,v}(z)$) for the generating functions of the $2n$-step transition probabilities to return to u (for the first time) after starting in u for the SRW on $B_{u,v}$. $G_{u,v}$ has radius of convergence $R_{u,v} = R(B_{u,v})$; in general $R_{u,v}$ depends on the branch $B_{u,v}$.

PROPOSITION 3.1. *If the SRW on T is recurrent then the SRW on each branch of T is recurrent (null recurrent by Thm. 2.2).*

PROOF. If the SRW on T starts in u, the first step has to be to one of the adjacent



vertices $v, \ldots, w$, say to v. But then the first return to u (if ever) can only occur if the SRW on $B_{u,v}$ returns to u (T is a tree!). Therefore (see e.g. [8])

$$F_u(z) = \frac{1}{d(u)} F_{u,v}(z) + \cdots + \frac{1}{d(u)} F_{u,w}(z).$$

The recurrence of the SRW on T implies $F_u(1) = 1$. Since $F_{u,v}(1) \leq 1, \ldots, F_{u,w}(1) \leq 1$ we clearly get

$$F_{u,v}(1) = 1, \ldots, F_{u,w}(1) = 1;$$

so the SRW on each branch of T is recurrent.

Example 2.4 shows that Lemma 2.1 does not hold if recurrent is replaced by R-recurrent (the tree T_b is R-recurrent, but its branch $B_{p,r}$ is R-transient by Example 2.4.a).

Therefore we will exclude this possibility and make the following assumption:

(e) Neither the SRW on T nor that on any of its branches is R-positive recurrent (R depends of course on the SRW under consideration).

THEOREM 3.2. *Let $T = (V, E)$ have properties (a), (b), (c) and (e). Then*
(1) *$R(T) \leq R_{u,v}$ for all branches $B_{u,v}$ of T,*
(2) *to every vertex u there exists an adjacent vertex v such that $R(T) = R_{u,v}$,*
(3) *if $[u, v] \in E$ then $R_{u,v} = \min_{w \neq u} R_{v,w}$*
(*the* min *is taken over all vertices $w \neq u$, adjacent to v*).

PROOF. Let $u \in V$ be a vertex of T. Since all the coefficients in the generating function $G_u(z)$ are positive a well known result of *Pringsheim* tells us that $R = R(T)$ is the smallest positive singularity of $G_u(z)$. Since $f^n(u, u) \leq p^n(u, u)$ the radius of convergence of $F_u(z)$ is at least R.

Therefore if the SRW on T is R-transient then $F_u(R) < 1$. But $G_u(z) = (1 - F_u(z))^{-1}$ has $z = R$ as a singularity and so $z = R$ is a singularity of $F_u(z)$ too.

If the SRW on T is R-null recurrent then $F_u(R) = 1$ and $F'_u(R) = \infty$. So $z = R$ is a singularity of $F_u(z)$ too.

Up to now we have shown that under the assumptions of Theorem 3.2 the functions $G_u(z)$ and $F_u(z)$ have the same radius of convergence $R = R(T)$ and $F_u(R) \leq 1$.

The same argument shows that for every branch $B_{u,v}$ the functions $G_{u,v}(z)$ and $F_{u,v}(z)$ have the same radius of convergence $R_{u,v}$ and $F_{u,v}(R_{u,v}) \leq 1$.

From the proof of Proposition 3.1 we have

$$F_u(z) = \frac{1}{d(u)} \sum_{[u,v]\in E} F_{u,v}(z).$$

Therefore $R(T)$ = radius of convergence of $F_u = \min_{[u,v]\in E} R_{u,v}$ (all power series have positive coefficients). This proves (1) and (2).

Now consider part (3). A simple flow-graph analysis (see e.g. [8]) yields

$$F_{u,v}(z) = \frac{z}{d(v)} \left(1 - \frac{1}{d(v)} \sum_{\substack{[v,w]\in E \\ w\neq u}} F_{v,w}(z) \right)^{-1}.$$

As before, $R_{u,v}$ = radius of convergence of $F_{u,v} = \min_{\substack{[v,w]\in E \\ w\neq u}} R_{v,w}$, since by (1) and (2) applied to $T = B_{u,v}$ we have $R_{u,v} \leq R_{v,w}$.

Now we choose some vertex o of T as origin and walk away from o according to the induced orientation $\overset{o}{\to}$. We can assign to each positively oriented edge $[u, v]$ (i.e. $u \overset{o}{\to} v$ holds) the real number $R_{u,v} \geq 1$, the radius of convergence (of the generating function $G_{u,v}$) of the SRW on the branch $B_{u,v}$. If we walk away from o along any path these numbers $R_{u,v}$ form a non decreasing sequence. Furthermore we have

COROLLARY 3.3. *Let T be a tree with (a), (b), (c), (e).*
(1) *If $b = [b_1, b_2, b_3, \ldots]$ is any end of T, the following limit exists:*

$$R(b) = \lim_{n\to\infty} R_{b_n,b_{n+1}} \leq \infty$$

(*= radius of convergence of the end b*). *$R(b)$ is independent of the representation of b (as infinite path) and independent of the choice of* o.
(2) $R(T) = \min\{R(b) \mid b \in B(T)\}$
(3) *If T has property (d) then*

$$R(b) \leq D, \qquad \forall b \in B(T).$$

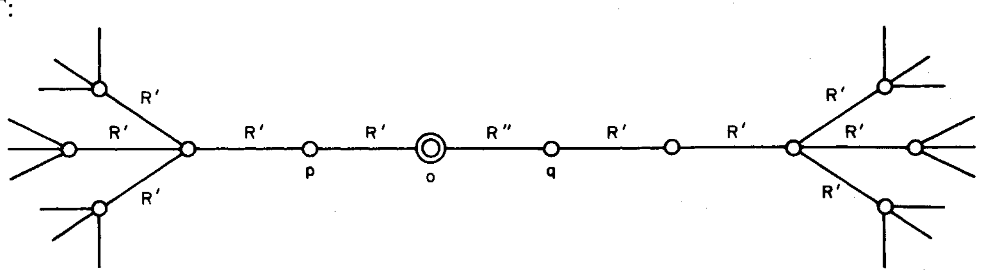
PROOF. We have only to prove (3): Let $[u, v]$ be any edge of T. We apply Lemma 2.1 to the SRW on $B_{u,v}$. This gives

$$\frac{1}{R_{u,v}} \geq \frac{p^2(u,u)}{p^0(u,u)} = p^2(u,u) = \frac{1}{d(v)} \geq \frac{1}{D}.$$

The next Example shows that Theorem 3.2 and Corollary 3.3 do not hold without property (e).

EXAMPLE 3.4. Consider the following tree T (a mixture of the trees T_b and T_c from Example 2.4):

T:



Then we have

$$F_{o,p}(z) = 3z(4+\sqrt{4-3z})^{-1} \qquad \text{(from Example 2.4.(b))},$$

$$F_{o,q}(z) = \frac{z}{2-F_{o,p}(z)}, \qquad F_o(z) = \tfrac{1}{2}(F_{o,p}(z) + F_{o,q}(z)).$$

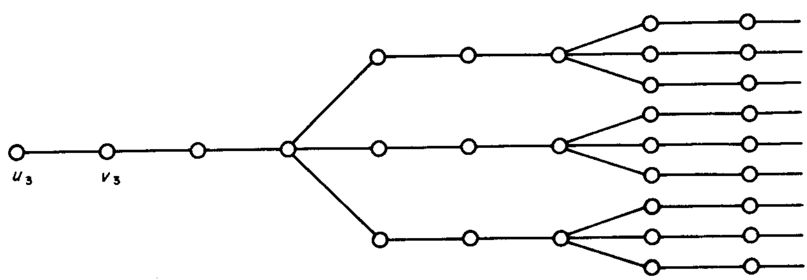
This implies that $R' = R_{o,p} = 4/3$, $B_{o,p}$ is R'-null recurrent, $R'' = R_{o,q}$ is the smallest positive solution of $F_{o,q}(z) = 1$, $1 < R'' < R'$, $B_{o,q}$ is R''-positive recurrent (as in Example 2.4). But $R = R(T)$ is the smallest positive solution of $F_o(z) = 1$, which implies $1 < R'' < R < R'$ and T is R-positive recurrent.

In the above drawing we wrote down to each edge $[u, v]$, pointing away from (the origin) o, the numbers $R_{u,v}$; it is interesting to note the $R = R(T)$ does not appear anywhere.

EXAMPLE 3.5. Consider trees T_p $(p = 1, 2, \ldots)$, with root vertex u_p and root edge $[u_p, v_p]$ defined as follows: All vertices at distance n from u_p have the same degree d_n where

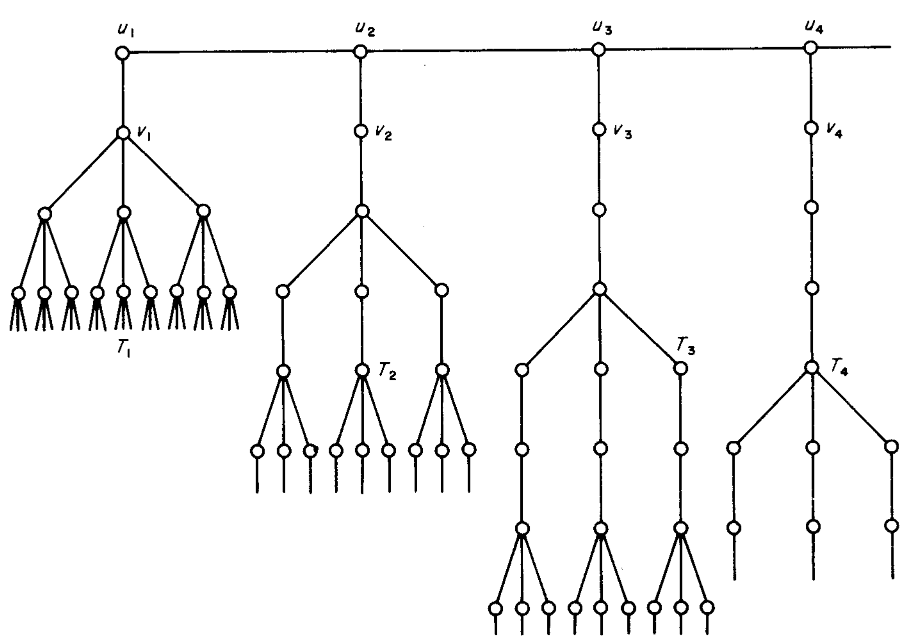
$$d_n = \begin{cases} 1 & \text{for } n = 0, \\ 4 & \text{if } p \text{ divides } n \geq 1, \\ 2 & \text{otherwise.} \end{cases}$$

T_3:



Now let $[u_1, u_2, \ldots]$ be an infinite path (half line) and attach at each vertex u_p the tree T_p; this gives the tree T we will study now:

T:



From [14] we have that the SRW on every T_p $(p=1,2,\ldots)$ is R_p-transient with

$$R_p = R(T_p) = \cos^{-2}\Phi_p, \qquad \Phi_p = \frac{\pi}{6p}.$$

We now verify property (e) for T:

(1) $R(T)$ is less than or equal to the radius of convergence of $F_{u_p}(z)$ and the latter is not greater than the radius of convergence of $F_{u_p,v_p}(z)$, which is equal to R_p. Therefore

$$1 \le R(T) \le R_p \to 1, \qquad \text{for } p \to \infty,$$

and $R(T)=1$. Proposition 3.1 implies that the SRW on T is transient (because the SRW on T_p is transient).

(2) Let v be a vertex of T_p, $v \neq u_p$. Then there is one branch $B_{v,w}$ at v containing the vertices $u_1, u_2, u_3, \ldots$. As in (1) $R(B_{v,w})=1$ and $B_{v,w}$ is transient. All other branches at v are R_p-transient (since they belong to T_p).

(3) Finally we consider the branches at u_p. One branch is T_p which is R_p-transient, another one is $B_{u_p,u_{p+1}}$ which is transient (as in (1)). For $p=1$ we are done. If $p=2$, F_{u_2,u_1} is just the function F_0 in Example 2.4.(b) and therefore B_{u_2,u_1} is R_1-null recurrent $(R_1=4/3)$. For $p \ge 2$ write $E_p(z)=F_{u_p,u_{p-1}}(z)$. Then

$$E_{p+1}(z)=\frac{z}{3}\left(1-\tfrac{1}{3}E_p(z)-\tfrac{1}{3}F_{u_p,v_p}(z)\right)^{-1}.$$

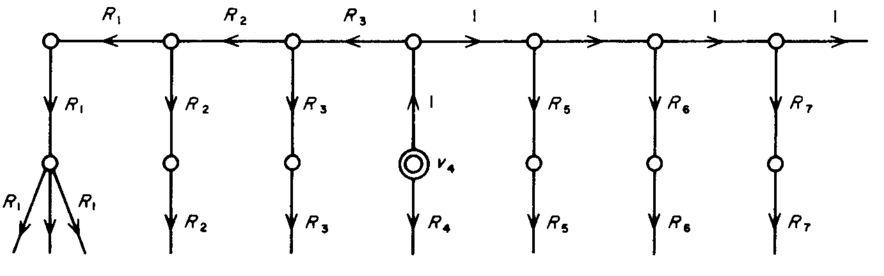
If we know that $E_p(z)$ has radius of convergence R_{p-1} and that $E_p(R_{p-1}) \le 1$ then, as $R_p < R_{p-1}$, we infer that $E_{p+1}(z)$ has the same radius of convergence as $F_{u_p,v_p}(z)$, namely R_p and [14]

$$F_{u_p,v_p}(R_p)=\frac{2}{\sqrt{3}}\tan\Phi_p$$

$$E_{p+1}(R_p) \le R_p\left(2-\frac{2}{\sqrt{3}}\tan\Phi_p\right)^{-1} < 1$$

(Since $2\Phi_p < \pi/3$). Therefore (by induction) the SRW on $B_{u_p,u_{p-1}}$ is R_{p-1}-transient for $p \ge 2$.

In the following figure, v_4 is chosen as origin and to each edge $[v, w]$ with $v \xrightarrow{v_4} w$ we associate $R_{v,w}$.



4. Growth

As usual let the distance $d(v, w)$ of two vertices v, w of any tree (graph) be defined by

$$d(v, w) = \text{number of edges of the shortest path between } v \text{ and } w.$$

Further write for $v \in V$, $n \in \mathbb{N}$

$$B(v, n)=\{w \in V \mid d(v, w) \le n\}, \quad \gamma(v, n)=|B(v, n)|$$

($|M|$ = cardinal number of the set M). The function $\gamma(v, n)$ describes the growth of the tree (graph) around v. In the following theorem the SRW and the growth are related (compare with [1] for symmetric random walks on groups).

THEOREM 4.1. *If the tree T has properties (a), (b) and (d), then $\gamma(v, n) \geq (1/D)R^n$, where $R = R(T)$.*

PROOF. Let $\Phi(z) = -z \log z$. Then $\Phi(z)$ is concave on the interval $[0, 1]$. Write for $v \in V$, $n \in \mathbb{N}$

$$A(v, n) = \{w \in V \mid p^n(v, w) > 0\} \text{ and } \alpha(v, n) = |A(v, n)|.$$

Then *Jensen's* inequality yields

$$\begin{aligned}\sum_w \Phi(p^n(v, w)) &\leq \alpha(v, n)\Phi\left(\sum_w \frac{1}{\alpha(v, n)} p^n(v, w)\right)\\ &= \alpha(v, n)\Phi\left(\frac{1}{\alpha(v, n)}\right) = \log \alpha(v, n).\end{aligned}$$

On the other hand, $z \mapsto \log z$ is concave on $[0, \infty)$, so again *Jensen's* inequality gives (since $\sum_w p^n(v, w) = 1$).

$$\begin{aligned}\sum_w p^n(v, w) \log p^n(v, w) &\leq \log \sum_w p^n(v, w) p^n(v, w)\\ &= \log \sum_w p^n(v, w) p^n(w, v) \frac{d(w)}{d(v)}\\ &\leq \log\left(D \sum_w p^n(v, w) p^n(w, v)\right) = \log(Dp^{2n}(v, v)),\end{aligned}$$

since by reversibility of the SRW: $d(v)p^n(v, w) = d(w)p^n(w, v)$. Combining these two inequalities we therefore get

$$Dp^{2n}(v, v) \geq \frac{1}{\alpha(v, n)}.$$

Clearly $\alpha(v, n) \leq \gamma(v, n)$. Now by Lemma 2.1 we have

$$\sqrt[n]{p^{2n}(v, v)} = \left(\frac{p^{2n}(v, v)}{p^{2n-2}(v, v)} \cdots \frac{p^4(v, v)}{p^2(v, v)} \cdot \frac{p^2(v, v)}{p^0(v, v)}\right)^{1/n} \leq \frac{p^{2n}(v, v)}{p^{2n-2}(v, v)} \leq \frac{1}{R},$$

therefore $p^{2n}(v, v) \leq R^{-n}$ and the proof is completed.

REMARK 4.2. Theorem 4.1 is true under much more general assumptions: If $p(v, w)$ are the transition probabilities of a random walk (Markov chain) on a locally finite, connected graph $G = (V, E)$, such that
(a) $p(v, w) > 0$ implies $v = w$ or $[v, w] \in E$,
(b) the chain is reversible, i.e. $d(v)p(v, w) = d(w)p(w, v)$ for positive numbers $d(v)$,
(c) $0 < m \leq d(v) \leq M < \infty$ for all $v \in V$,
then

$$\gamma(v, n) \geq \frac{m}{M} R^n, \qquad \text{for all } v \in V,$$

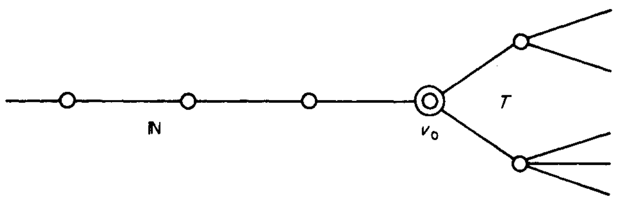
where R and $\gamma(v, n)$ are defined as above.

For our next example we need the following simple but perhaps surprising proposition:

PROPOSITION 4.3. *Let T be any tree, v_0 a vertex of T. Attach a half-line $\mathbb{N}$ to T at v_0 to give a tree $T^{\mathbb{N}}$. Then the radius of convergence of (the SRW on) $T^{\mathbb{N}}$*

$$R(T^{\mathbb{N}}) = 1.$$

$T^{\mathbb{N}}$:



PROOF. Write $F_{v_0}(F^{\mathbb{N}}_{v_0})$ for the generating functions of the first return probabilities to v_0 in T (resp. $T^{\mathbb{N}}$) and let d_0 be the degree of the vertex v_0 in T. Then

$$F^{\mathbb{N}}_{v_0}(z) = \frac{1}{d_0+1}(1-\sqrt{1-z}) + \frac{d_0}{d_0+1} F_{v_0}(z).$$

Therefore $F^{\mathbb{N}}_{v_0}$ has $z=1$ as its smallest positive singularity implying $R(T^{\mathbb{N}})=1$.

Theorem 4.1 implies in particular that a tree T with $R(T)>1$ has exponential growth (which is of course independent of the vertex v). However, the following example shows that $R(T)=1$ may occur even if T has (fast) exponential growth.

EXAMPLE 4.4. Let T be any tree with $R(T)>1$ (so T has exponential growth) and v_0 a vertex of T. Attach a half-line $\mathbb{N}$ to T at v_0 to give a tree $T^{\mathbb{N}}$. Then by the proposition above $R(T^{\mathbb{N}})=1$, but $T^{\mathbb{N}}$ has exponential growth too. We can be more precise, using *Darboux's* method to get

$$p^{2n}(v_0, v_0) \underset{(n\to\infty)}{\sim} c \cdot n^{-3/2} \quad (\text{in } T^{\mathbb{N}}).$$

We see that $T^{\mathbb{N}}$ has inherited
(1) transience from T and
(2) $R(T^{\mathbb{N}})=1$ from the half-line $\mathbb{N}$.

If T has properties (a)-(e) then we can get more information when we consider the whole spectrum of numbers $R(b)$, $b \in B(T)$.

COROLLARY 4.5. *Let T be a tree with properties (a)-(e). If $\varepsilon>0$ is arbitrary and*

$$S = \sup\{R(b) \mid b \in B(T)\}$$

then there is a vertex $v = v(\varepsilon)$ such that

$$\gamma(v, n) \geq \frac{1}{D}(S-\varepsilon)^n.$$

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NOTE ADDED IN PROOF

Since this paper was submitted several interesting related contributions have appeared among which we would like to point out.

J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, *Trans. Amer. Math. Soc.* **284** (1984), 787-794.

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